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On Convergence to Equilibrium Distribution, II. The Wave Equation in Odd Dimensions, with Mixing

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The paper considers the wave equation, with constant or variable coefficients in $\mathbb{R}^n$, with odd $n \geqslant 3$. We study the asymptotics of the distribution μ_t of the *random solution* at time $t \in \mathbb{R}$ as $t \to \infty$. It is assumed that the initial measure μ_0 has zero mean, translation-invariant covariance matrices, and finite expected energy density. We also assume that μ_0 satisfies a Rosenblatt- or Ibragimov–Linnik-type space mixing condition. The main result is the convergence of μ_t to a Gaussian measure μ_∞ as $t \to \infty$, which gives a Central Limit Theorem (CLT) for the wave equation. The proof for the case of constant coefficients is based on an analysis of long-time asymptotics of the solution in the Fourier representation and Bernstein's "room-corridor" argument. The case of variable coefficients is treated by using a version of the scattering theory for infinite energy solutions, based on Vainberg's results on local energy decay.

KEY WORDS: Wave equation; Cauchy problem; random initial data; mixing condition; Fourier transform; convergence to a Gaussian measure; covariance functions and matrices.

1. INTRODUCTION

This paper can be considered as a continuation of ref. 1. Here we develop a probabilistic analysis for the linear wave equation (WE) in $\mathbb{R}^n$, with odd $n \geqslant 3$:

Dedicated to Ya. G. Sinai on the occasion of his 65th anniversary.

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$$\begin{cases} \ddot{u}(x,t) = Lu(x,t) \equiv \sum\limits_{i,j=1}^{n} \partial_i(a_{ij}(x)\,\partial_j u(x,t)) - a_0(x)\,u(x,t), \\ u|_{t=0} = u_0(x), \qquad \dot{u}|_{t=0} = v_0(x), \end{cases} \tag{1.1}$$

where $\partial_i \equiv \frac{\partial}{\partial x_i}$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$. We assume that the coefficients of equation (1.1) are constant outside a bounded region, more precisely, $a_{ij}(x) = \delta_{ij}$ for $|x| \geqslant \text{const}$. Moreover, we assume that a *nontrapping* condition is satisfied, i.e., all rays of a geometrical optics associated with (1.1) go to infinity (see Condition E3 in Section 2.1). Denote $Y(t) = (Y^0(t), Y^1(t)) \equiv (u(\cdot, t), \dot{u}(\cdot, t))$, $Y_0 = (Y_0^0, Y_0^1) \equiv (u_0, v_0)$. Then (1.1) becomes

$$\dot{Y}(t) = \mathscr{A} Y(t), \quad t \in \mathbb{R}; \qquad Y(0) = Y_0. \tag{1.2}$$

Here we set:

$$\mathscr{A} = \begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix}, \tag{1.3}$$

where $A = \sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\,\partial_j) - a_0(x)$. We suppose that the initial date Y_0 is a random function with zero mean living in a functional phase space $\mathscr{H}$ representing states with finite local energy; the distribution of Y_0 is denoted by μ_0. Denote by μ_t, $t \in \mathbb{R}$, the measure on $\mathscr{H}$ giving the distribution of the random solution $Y(t)$ to problem (1.2). We assume that the initial covariance functions (CFs) are translation-invariant, i.e.

$$Q_0^{ij}(x,y) := E(Y_0^i(x)\,Y_0^j(y)) = q_0^{ij}(x-y), \qquad x, y \in \mathbb{R}^n, \quad i, j = 0, 1. \tag{1.4}$$

Next, we assume that the initial "mean energy density" is finite:

$$\begin{aligned} e_0 &:= E(|v_0(x)|^2 + |\nabla u_0(x)|^2 + |u_0(x)|^2) \\ &= q_0^{11}(0) - \Delta q_0^{00}(0) + q_0^{00}(0) < \infty, \qquad x \in \mathbb{R}^n. \end{aligned} \tag{1.5}$$

Finally, it is assumed that μ_0 satisfies a space-mixing condition. Roughly speaking, it means that

$$Y_0(x) \text{ and } Y_0(y) \text{ are asymptotically independent as } |x-y| \to \infty. \tag{1.6}$$

Our main result establishes the convergence

$$\mu_t \rightharpoondown \mu_\infty, \qquad t \to \infty. \tag{1.7}$$

to a stationary measure μ_∞ that is Gaussian and supported in space $\mathscr{H}$.

Previously, results of this kind have been obtained in refs. 2–5, for translation-invariant initial measures μ_0. However, the detailed proofs were not published because of their length. Another drawback was the absence of a unifying argument indicating the limits of the methods. In this paper, such an argument is presented, which also improves the assumptions and makes the proofs much shorter. Like ref. 1, the argument is based on a systematic use of Fourier transform (FT) and a duality argument in Lemma 5.1 (cf. refs. 6–8 concerning FT arguments for lattice systems). This is used in conjunction with the *strong* Huyghen's principle for the WE and the fact that the rank of the Hessian of the restricted dispersion relation (5.20) equals $n-1$. We also found a more efficient method to estimate higher order momentum functions and to complete some details in the proof of scattering theory results for the case of variable coefficients. The mixing condition has been used in refs. 6–9 to prove the convergence for various classes of systems. In this paper it is used in the context of the WE.

We prove relation (1.7) by using the strategy similar to ref. 1. At first, we prove (1.7) for the equations with constant coefficients $a_{ij}(x) \equiv \delta_{ij}$, in three steps.

I. The family of measures μ_t, $t \geqslant 0$, is compact in an appropriate Fréchet space.

II. The CFs converge to a limit: for $i, j = 0, 1$,

$$Q_t^{ij}(x, y) = \int Y^i(x)\, Y^j(y)\, \mu_t(dY) \to Q_\infty^{ij}(x, y), \qquad t \to \infty. \tag{1.8}$$

III. The characteristic functionals converge to Gaussian:

$$\hat{\mu}_t(\Psi) = \int \exp(i\langle Y, \Psi\rangle)\, \mu_t(dY) \to \exp\{-\tfrac{1}{2}\,\mathcal{Q}_\infty(\Psi, \Psi)\}, \qquad t \to \infty, \tag{1.9}$$

where Ψ is an arbitrary element of a dual space and $\mathcal{Q}_\infty$ is the quadratic form with the integral kernel $(Q_\infty^{ij}(x, y))_{i, j=0, 1}$.

Property I follows from Prokhorov's Compactness Theorem with the help of arguments from ref. 10. First, one proves a uniform bound for the mean local energy in measure μ_t with the help of the FT. The conditions of Prokhorov' Theorem the follow from Sobolev's Embedding Theorem. Property II is deduced from an analysis of oscillatory integrals arising in the FT. An important role is attributed to Proposition 4.1 which establishes useful properties of the CFs in the FT deduced from the mixing condition.

On the other hand, the FT alone is not sufficient to prove Property III even in the case of constant coefficients. The reason is that a function of infinite energy gives a singular generalised function in the FT, and an exact interpretation of condition (1.6) in these terms is unclear. We deduce Property III from a representation for the solution in the coordinate space, which manifests a dispersion of waves. In particular, for the case $n=3$ and $u_0(x)\equiv 0$, Kirchhoff's formula holds:

$$u(x,t)=\frac{1}{4\pi t}\int_{S_t(x)} v_0(y)\,dS(y), \qquad x\in\mathbb{R}^3, \tag{1.10}$$

where $dS(y)$ is the Lebesgue measure on the sphere $S_t(x)$: $|y-x|=t$. Then the proof of (1.9) proceeds with a modification of Bernstein's "room-corridor" method, well-known in the random processes theory. Namely, we divide the sphere of integration in (1.10) into "rooms" R_t^k, $1\leqslant k\leqslant N$, of a fixed width $d\gg 1$, separated by "corridors" C_t^k of a fixed width $\rho\ll d$. As the area $|R_t^k|\sim d^2=\text{const}$, the number $N\sim t^2$, and (1.10) becomes

$$u(x,t)\sim\frac{\sum_{k=1}^N r_t^k}{\sqrt{N}}, \tag{1.11}$$

where r_t^k is the integral over R_t^k. The contribution of the "corridors" turns out to be negligible. Assume for a moment that $v_0(x)$ and $v_0(y)$ are independent for $|x-y|\geqslant\rho_0$. Then r_t^k are independent if $\rho>\rho_0$, and random variable $u(x,t)$ is asymptotically Gaussian by the CLT.

So, the CLT emerges from (1.10) because of integration over the sphere $|y-x|=t$ and the first power of t in the denominator. A similar geometrical structure of an integral over the sphere $|y-x|=t$ emerges from Herglotz–Petrovskii's formulas in a general odd dimension $n\geqslant 5$. However, the extension of the argument based on (1.10), (1.11) is not straightforward for $n\geqslant 5$ as the Herglotz–Petrovskii's formulas contain high-order derivatives of initial functions.

We cover all odd values $n\geqslant 3$ in a unified techniques by modifying the approach developed in ref. 1 for the Klein–Gordon equation (KGE). However, for the KGE, the solution is an integral over the ball $|y-x|\leqslant t$. This fact allowed us to use for the KGE a rather different approach based on the analysis of an oscillatory integral where the phase function ("dispersion relation") has a nondegenerate Hessian. For the WE, the Hessian is degenerate, which requires additional constructions. Here we use the fact that the "restricted" Hessian has a maximal rank $n-1$, see (5.20). This leads to a weaker dispersion of waves comparing to the KGE. Newertheless, we still obtain the representation of the solution as a sum of weakly dependent random variables. Then (1.9) follows from the CLT. However, checking the Lindeberg condition for the WE requires some delicate calculations. Here, the deficiancy in dispersive properties is compensated by the reduction in dimension of the domain of integration due to the strong Huyghen's principle.

All three steps I–III of our argument rely on the mixing condition. Simple examples show that the convergence to a Gaussian measure may fail when the mixing condition fails: if we take $u_0(x) \equiv 0$ and $v_0(x) \equiv \pm 1$ with probability $p_\pm = 0.5$, then $u(x, t) \equiv \pm t$ almost sure.

Finally, we prove the convergence in (1.7) for problem (1.1) with variable coefficients. In this case explicit formulas for the solution are unavailable. To prove (1.7) in this case, we use a version of the scattering theory for solutions of infinite global energy (this strategy is similar to ref. 11). This allows us to reduce the proof to the case of constant coefficients. Namely, we establish the long-time asymptotics

$$U(t)\, Y_0 = \Theta U_0(t)\, Y_0 + \rho(t)\, Y_0, \qquad t > 0, \tag{1.12}$$

where $U(t)$ is the dynamical group of Eq. (1.1), $U_0(t)$ corresponds to the constant coefficients $a_{jk}(x) \equiv \delta_{ij}$, and Θ is a "scattering operator." The remainder, $\rho(t)$, is small in local energy seminorms $\|\cdot\|_R$, $\forall R > 0$:

$$\|\rho(t)\, Y_0\|_R \to 0, \qquad t \to \infty. \tag{1.13}$$

The scattering theory results are based on the Vainberg's estimates for the local energy decay; see ref. 12.

Remark 1.1. (i) Under our assumptions on initial measure μ_0, initial date Y_0 has an infinite energy. Therefore, the standard scattering theory, for the solutions of a finite energy (see, e.g., ref. 13), is not sufficient for our purposes.

(ii) The order of the operators in product $\Theta U_0(t)$ in (1.12) differs from that in $U_0(t)\, \Theta$ considered in the scattering theory of finite energy solutions. An asymptotics with the order $U_0(t)\, \Theta$ would mean that $Y(t)$ is close to a solution of the unperturbed equation. This is impossible for the solutions of infinite energy as they do not converge locally to zero, hence the perturbation terms in the equation are not negligible.

The paper is organised as follows. In Section 2 we formally state our main result. Sections 3–7 deal with the case of constant coefficients: main results are stated in Section 3, the compactness (Property I) and the convergence (1.8) are proved in Section 4, and convergence (1.9) in Sections 5 and 6. In Section 7 we check the Lindeberg condition.

In Section 8 we construct the scattering theory, and in Section 9 establish convergence (1.7) for variable coefficients. Section 10 discusses Vainberg's estimates. In the Appendix we collected the FT calculations.

2. MAIN RESULTS

2.1. The Notation

Denote by D the space of real functions $C_0^\infty(\mathbb{R}^n)$. We assume that the following properties E1–E3 of Eq. (1.1) are satisfied:

E1 $a_{ij}(x) = \delta_{ij} + b_{ij}(x)$, where $b_{ij}(x) \in D$; also $a_0(x) \in D$.

E2 $a_0(x) \geqslant 0$, and the hyperbolicity condition holds: $\exists \alpha > 0$

$$H(x, k) \equiv \tfrac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x)\, k_i k_j \geqslant \alpha\, |k|^2, \qquad x, k \in \mathbb{R}^n. \tag{2.1}$$

E3 A non-trapping condition:[12] for $(x(0), k(0)) \in \mathbb{R}^n \times \mathbb{R}^n$ with $k(0) \neq 0$,

$$|x(t)| \to \infty \qquad \text{as} \quad t \to \infty, \tag{2.2}$$

where $(x(t), k(t))$ is a solution to the Hamiltonian system

$$\dot{x}(t) = \nabla_k H(x(t), k(t)), \qquad \dot{k}(t) = -\nabla_x H(x(t), k(t)).$$

Example. E1–E3 hold for the acoustic equation with constant coefficients

$$\ddot{u}(x, t) = \Delta u(x, t), \qquad x \in \mathbb{R}^n. \tag{2.3}$$

For instance, E3 follows because $\dot{k}(t) \equiv 0 \Rightarrow x(t) \equiv k(0)\, t + x(0)$.

We assume that the initial date Y_0 belongs to the phase space $\mathscr{H}$ defined below.

Definition 2.1. $\mathscr{H} \equiv H^1_{\rm loc}(\mathbb{R}^n) \oplus H^0_{\rm loc}(\mathbb{R}^n)$ is the Fréchet space of pairs $Y(x) \equiv (u(x), v(x))$ of real functions $u(x)$, $v(x)$, endowed with local energy seminorms

$$\|Y\|_R^2 = \int_{|x|<R} (|v(x)|^2 + |\nabla u(x)|^2 + |u(x)|^2)\, dx < \infty, \qquad \forall R > 0. \tag{2.4}$$

Proposition 2.2 follows from ref. 14, Thms. V.3.1 and V.3.2) as the speed of propagation for Eq. (1.1) is finite.

Proposition 2.2. (i) For any $Y_0 \in \mathcal{H}$ there exists a unique solution $Y(t) \in C(\mathbb{R}, \mathcal{H})$ to Cauchy problem (1.2).

(ii) For any $t \in \mathbb{R}$, the operator $U(t)$: $Y_0 \mapsto Y(t)$ is continuous in $\mathcal{H}$.

We now introduce appropriate Hilbert spaces of initial data of infinite energy. Let δ be an arbitrary positive number.

Definition 2.3. $\mathcal{H}_\delta$ is the Hilbert space of the functions $Y = (u, v) \in \mathcal{H}$ with a finite norm

$$|||Y|||_\delta^2 = \int e^{-2\delta |x|}(|v(x)|^2 + |\nabla u(x)|^2 + |u(x)|^2)\, dx < \infty.$$

Let us choose a function $\zeta(x) \in C_0^\infty(\mathbb{R}^n)$ with $\zeta(0) \neq 0$. Denote by $H^s_{\rm loc}(\mathbb{R}^n)$, $s \in \mathbb{R}$, the local Sobolev spaces, i.e., the Fréchet spaces of distributions $u \in D'(\mathbb{R}^n)$ with finite seminorms

$$\|u\|_{s,R} := \|\Lambda^s(\zeta(x/R)\, u)\|_{L^2(\mathbb{R}^n)},$$

where $\Lambda^s v := F^{-1}_{k \to x}(\langle k \rangle^s\, \hat{v}(k))$, $\langle k \rangle := \sqrt{|k|^2 + 1}$, and $\hat{v} := Fv$ is the FT of a tempered distribution v. For $\psi \in D$ define $F\psi(k) = \int e^{ik \cdot x}\psi(x)\, dx$.

Definition 2.4. For $s \in \mathbb{R}$ denote $\mathcal{H}^s \equiv H^{1+s}_{\rm loc}(\mathbb{R}^n) \oplus H^s_{\rm loc}(\mathbb{R}^n)$.

Using standard techniques of pseudodifferential operators and Sobolev's Embedding Theorem (see, e.g., ref. 15), it is possible to prove that $\mathcal{H}^0 = \mathcal{H} \subset \mathcal{H}^{-\varepsilon}$ for every $\varepsilon > 0$, and the embedding is compact. We denote by $\langle \cdot, \cdot \rangle$ scalar product in real Hilbert space $L^2(\mathbb{R}^n)$ or in $L^2(\mathbb{R}^n) \otimes \mathbb{R}^N$ or its various extensions.

2.2. Random Solution. Convergence to Equilibrium

Let (Ω, Σ, P) be a probability space with expectation E and $\mathcal{B}(\mathcal{H})$ denote the Borel σ-algebra in $\mathcal{H}$. We assume that $Y_0 = Y_0(\omega, x)$ in (1.2) is a measurable random function with values in $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$. In other words, $(\omega, x) \mapsto Y_0(\omega, x)$ is a measurable map $\Omega \times \mathbb{R}^n \to \mathbb{R}^2$ with respect to the (completed) σ-algebras $\Sigma \times \mathcal{B}(\mathbb{R}^n)$ and $\mathcal{B}(\mathbb{R}^2)$. Then $Y(t) = U(t)\, Y_0$ is also a measurable random function with values in $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$ owing to Proposition 2.2. We denote by $\mu_0(dY_0)$ a Borel probability measure in $\mathcal{H}$ giving the distribution of the Y_0. Without loss of generality, we assume $(\Omega, \Sigma, P) = (\mathcal{H}, \mathcal{B}(\mathcal{H}), \mu_0)$ and $Y_0(\omega, x) = \omega(x)$ for $\mu_0(d\omega) \times dx$-almost all $(\omega, x) \in \mathcal{H} \times \mathbb{R}^n$.

Definition 2.5. μ_t is a Borel probability measure in $\mathcal{H}$ which gives the distribution of $Y(t)$:

$$\mu_t(B) = \mu_0(U(-t)\, B), \qquad \forall B \in \mathcal{B}(\mathcal{H}), \quad t \in \mathbb{R}. \tag{2.5}$$

Our main goal is to derive the convergence of the measures μ_t as $t \to \infty$. We establish the weak convergence of μ_t in the Fréchet spaces $\mathcal{H}^{-\varepsilon}$ with any $\varepsilon > 0$:

$$\mu_t \overset{\mathcal{H}^{-\varepsilon}}{\rightharpoondown} \mu_\infty \qquad \text{as} \quad t \to \infty, \tag{2.6}$$

where μ_∞ is a Borel probability measure in space $\mathcal{H}$. By definition, this means the convergence

$$\int f(Y)\, \mu_t(dY) \to \int f(Y)\, \mu_\infty(dY) \qquad \text{as} \quad t \to \infty \tag{2.7}$$

for any bounded continuous functional $f(Y)$ in space $\mathcal{H}^{-\varepsilon}$.

Definition 2.6. The CFs of measure μ_t are defined by

$$Q_t^{ij}(x, y) \equiv E(Y^i(x, t)\, Y^j(y, t)), \quad i, j = 0, 1, \quad \text{for almost all } x, y \in \mathbb{R}^n \times \mathbb{R}^n \tag{2.8}$$

if the expectations in the RHS are finite.

We set $\mathcal{D} = D \oplus D$, and $\langle Y, \Psi \rangle = \langle Y^0, \Psi^0 \rangle + \langle Y^1, \Psi^1 \rangle$ for $Y = (Y^0, Y^1) \in \mathcal{H}$, and $\Psi = (\Psi^0, \Psi^1) \in \mathcal{D}$. For a Borel probability measure μ in the space $\mathcal{H}$ we denote by $\hat{\mu}$ the characteristic functional (the Fourier transform of μ)

$$\hat{\mu}(\Psi) \equiv \int \exp(i\langle Y, \Psi \rangle)\, \mu(dY), \qquad \Psi \in \mathcal{D}.$$

A measure μ is called Gaussian (with zero expectation) if its characteristic functional has the form

$$\hat{\mu}(\Psi) = \exp\{-\tfrac{1}{2}\, \mathcal{Q}(\Psi, \Psi)\}, \qquad \Psi \in \mathcal{D},$$

where $\mathscr{Q}$ is a real nonnegative quadratic form in $\mathscr{D}$. A measure μ is called translation-invariant if

$$\mu(T_h B) = \mu(B), \qquad \forall B \in \mathscr{B}(\mathscr{H}), \quad h \in \mathbb{R}^n,$$

where $T_h Y(x) = Y(x-h)$.

2.3. Mixing Condition

Let $O(r)$ denote the set of all pairs of open subsets $\mathscr{A}, \mathscr{B} \subset \mathbb{R}^n$ at distance $\rho(\mathscr{A}, \mathscr{B}) \geqslant r$ and $\sigma(\mathscr{A})$ be the σ-algebra of the subsets in $\mathscr{H}$ generated by all linear functionals $Y \mapsto \langle Y, \Psi \rangle$, where $\Psi \in \mathscr{D}$ with $\operatorname{supp} \Psi \subset \mathscr{A}$. We define the Ibragimov–Linnik mixing coefficient of a probability measure μ_0 on $\mathscr{H}$ by (cf. ref. 16, Dfn. 17.2.2)

$$\varphi(r) \equiv \sup_{(\mathscr{A}, \mathscr{B}) \in O(r)} \sup_{\substack{A \in \sigma(\mathscr{A}),\, B \in \sigma(\mathscr{B}) \\ \mu_0(B) > 0}} \frac{|\mu_0(A \cap B) - \mu_0(A)\, \mu_0(B)|}{\mu_0(B)}. \tag{2.9}$$

Definition 2.7. Measure μ_0 satisfies the strong uniform Ibragimov–Linnik mixing condition if

$$\varphi(r) \to 0, \qquad r \to \infty. \tag{2.10}$$

Below, we specify the rate of the decay.

2.4. Main Theorem

We assume that measure μ_0 satisfies the following properties S0–S3:

S0 μ_0 has the zero expectation value,

$$EY_0(x) \equiv 0, \qquad x \in \mathbb{R}^n. \tag{2.11}$$

S1 The CFs of μ_0 are translation invariant, i.e., Eq. (1.4) holds for almost all $x, y \in \mathbb{R}^n$.

S2 μ_0 has a finite "mean energy density," i.e., Eq. (1.5) holds.

S3 Measure μ_0 satisfies the strong uniform Ibragimov–Linnik mixing condition, and

$$\bar{\varphi} \equiv \int_0^\infty r^{n-2} \varphi^{1/2}(r)\, dr < \infty. \tag{2.12}$$

Remark 2.8. (1.5) implies that μ_0 is concentrated in $\mathscr{H}_\delta$ for all $\delta>0$, since

$$\int |||Y_0|||_\delta^2\,\mu_0(dY_0)=e_0\int \exp(-2\delta\,|x|)\,dx<\infty. \tag{2.13}$$

Let $\mathscr{E}(x)=-C_n\,|x|^{2-n}$ be the fundamental solution of the Laplacian, i.e., $\Delta\mathscr{E}(x)=\delta(x)$ for $x\in\mathbb{R}^n$. Define, for almost all $x,y\in\mathbb{R}^n$, the matrix-valued function

$$Q_\infty(x,y)=(Q_\infty^{ij}(x,y))_{i,j=0,1}=(q_\infty^{ij}(x-y))_{i,j=0,1}, \tag{2.14}$$

where

$$(q_\infty^{ij})_{i,j=0,1}=\frac{1}{2}\begin{pmatrix} q_0^{00}-\mathscr{E}*q_0^{11} & q_0^{01}-q_0^{10}\\ q_0^{10}-q_0^{01} & q_0^{11}-\Delta q_0^{00}\end{pmatrix}. \tag{2.15}$$

According to ref. 16, Lemma 17.2.3 (see Lemma 6.2(i) later), the derivatives $\partial_z^\alpha q_0^{ij}$ are bounded by mixing coefficient: $\forall\alpha\in\mathbb{Z}_+^n$ with $|\alpha|\leqslant 2-i-j$ (including $\alpha=0$), $i,j=0,1$,

$$|\partial_z^\alpha q_0^{ij}(z)|\leqslant 2e_0\varphi^{1/2}(|z|),\qquad \forall z\in\mathbb{R}^n. \tag{2.16}$$

Hence, (2.12) implies the existence of the convolution $\mathscr{E}*q_0^{11}$ in (2.15). Denote by $\mathscr{Q}_\infty(\Psi,\Psi)$ a real quadratic form in $\mathscr{D}$ defined by

$$\mathscr{Q}_\infty(\Psi,\Psi)=\sum_{i,j=0,1}\int_{\mathbb{R}^n\times\mathbb{R}^n} Q_\infty^{ij}(x,y)\,\Psi^i(x)\,\Psi^j(y)\,dx\,dy. \tag{2.17}$$

Our main result is the following theorem.

Theorem A. Let $n\geqslant 3$ be odd, and E1–E3, S0–S3 hold. Then

(i) The convergence in (2.6) holds for any $\varepsilon>0$.

(ii) The limiting measure μ_∞ is a Gaussian equilibrium measure on $\mathscr{H}$.

(iii) The limiting characteristic functional has the form

$$\hat\mu_\infty(\Psi)=\exp\{-\tfrac{1}{2}\,\mathscr{Q}_\infty(W\Psi,W\Psi)\},\qquad \Psi\in\mathscr{D},$$

where $W\colon\mathscr{D}\to\mathscr{H}'_\delta$ is a linear continuous operator for sufficiently small $\delta>0$, and the quadratic form $\mathscr{Q}_\infty$ is continuous in $\mathscr{H}'_\delta$.

2.5. Remarks on Various Mixing Conditions for Initial Measure

We use strong uniform Ibragimov–Linnik mixing condition for the simplicity of presentation. The *uniform* Rosenblatt mixing condition[17] also is sufficient together with a higher degree >2 in the bound (1.5): there exists $\delta > 0$ such that

$$\sup_{x \in \mathbb{R}^n} E(|v_0(x)|^{2+\delta} + |\nabla u_0(x)|^{2+\delta} + |u_0(x)|^{2+\delta}) < \infty. \tag{1.8'}$$

Then (2.12) requires a modification:

$$\int_0^\infty r^{n-2} \alpha^p(r)\, dr < \infty, \qquad p = \min\left(\frac{\delta}{2+\delta}, \frac{1}{2}\right), \tag{2.12'}$$

where $\alpha(r)$ is the Rosenblatt mixing coefficient defined as in (2.9) but without $\mu(B)$ in the denominator. The statements of Theorem A and their proofs remain essentially unchanged, only Lemma 6.2 requires a suitable modification.[16]

3. EQUATIONS WITH CONSTANT COEFFICIENTS

In Sections 3–7 we consider the Cauchy problem (1.1) with the constant coefficients, i.e.,

$$\begin{cases} \ddot{u}(x,t) = \Delta u(x,t), & x \in \mathbb{R}^n, \\ u|_{t=0} = u_0(x), & \dot{u}|_{t=0} = v_0(x). \end{cases} \tag{3.1}$$

Rewrite (3.1) in the form similar to (1.2):

$$\dot{Y}(t) = \mathscr{A}_0 Y(t), \quad t \in \mathbb{R}; \qquad Y(0) = Y_0. \tag{3.2}$$

Here we denote

$$\mathscr{A}_0 = \begin{pmatrix} 0 & 1 \\ A_0 & 0 \end{pmatrix}, \tag{3.3}$$

where $A_0 = \Delta$. Denote by $U_0(t)$, $t \in \mathbb{R}$, the dynamical group for problem (3.2), then $Y(t) = U_0(t)\, Y_0$. Set $\mu_t(B) = \mu_0(U_0(-t)\, B)$, $B \in \mathscr{B}(\mathscr{H})$, $t \in \mathbb{R}$. The main result for the problem (3.2) is the following

Theorem B. Let $n \geqslant 3$ be odd, and Conditions S0–S3 hold. Then the conclusions of Theorem A hold with $W = I$, and limiting measure μ_∞ is translation-invariant.

Theorem B can be deduced from Propositions 3.1 and 3.2 below, by using the same arguments as in ref. 10, Thm. XII.5.2.

Proposition 3.1. The family of measures $\{\mu_t, t \geqslant 0\}$ is weakly compact in $\mathcal{H}^{-\varepsilon}$ with any $\varepsilon > 0$, and the following bounds hold:

$$\sup_{t \geqslant 0} E \, \|U_0(t)\, Y_0\|_R^2 < \infty, \qquad R > 0. \tag{3.4}$$

Proposition 3.2. For any $\Psi \in \mathcal{D}$,

$$\hat{\mu}_t(\Psi) \equiv \int \exp(i\langle Y, \Psi\rangle)\, \mu_t(dY) \to \exp\{-\tfrac{1}{2}\,\mathcal{Q}_\infty(\Psi, \Psi)\}, \qquad t \to \infty. \tag{3.5}$$

Propositions 3.1 and 3.2 are proved in Sections 4 and 5–7, respectively. We will use repeatedly the FT formulas (A.2) and (A.5) from Appendix A.

4. COMPACTNESS OF THE MEASURE FAMILY

Here we prove Proposition 3.1 with the help of FT.

4.1. Mixing in Terms of the Fourier Transform

The next proposition reflects the mixing property in terms of the FT $\hat{q}_0^{ij}$ of initial CFs q_0^{ij}. Assumption S2 implies that $q_0^{ij}(z)$ is a measurable bounded function. Therefore, it belongs to the Schwartz space of tempered distributions as well as its FT.

Proposition 4.1. $\hat{q}_0^{ij}(k) \in L^p(\mathbb{R}^n)$ with $1 \leqslant p \leqslant 2$, and

$$\int (|k|^{2-i-j} + |k|^{-2})\, |\hat{q}_0^{ij}(k)|\, dk \leqslant C(\varphi)\, e_0 < \infty, \qquad i, j = 0, 1. \tag{4.1}$$

Proof. We check the bound for $i = j = 1$ (in all other cases the proof is similar). By the Bohner Theorem, $\hat{q}_0^{11}\, dk$ is a nonnegative measure. Hence,

$$\int |\hat{q}_0^{11}(k)\, dk| = \int \hat{q}_0^{11}(k)\, dk = q_0^{11}(0) < \infty, \tag{4.2}$$

owing to S2. Similarly, (2.16) and (2.12) imply that

$$\int |k|^{-2} |\hat{q}_0^{11}(k)\, dk| = \int |k|^{-2}\, \hat{q}_0^{11}(k)\, dk = C \int |x|^{2-n}\, q_0^{11}(x)\, dx$$

$$\leqslant C_1 e_0 \int_0^{+\infty} r\varphi^{1/2}(r)\, dr \leqslant C_2 \bar{\varphi} e_0 < \infty. \tag{4.3}$$

It remains to prove that measure $\hat{q}_0^{11}(k)\, dk$ is absolutely continuous with respect to the Lebesgue measure. Function $\phi(r)$ is nonincreasing, hence by (2.12)

$$r^{n-1}\varphi^{1/2}(r) = (n-1) \int_0^r s^{n-2}\varphi^{1/2}(r)\, ds \leqslant (n-1) \int_0^r s^{n-2}\varphi^{1/2}(s)\, ds \leqslant C\bar{\varphi} < \infty. \tag{4.4}$$

Then (2.16) and (2.12) imply

$$\int |q_0^{11}(z)|^2\, dz \leqslant 4e_0^2 \int_{\mathbb{R}^n} \varphi(|z|)\, dz = C(n)\, e_0^2 \int_0^\infty r^{n-1}\varphi(r)\, dr$$

$$\leqslant C_1 \bar{\varphi} e_0^2 \int_0^\infty \varphi^{1/2}(r)\, dr < \infty.$$

Therefore, $\hat{q}_0^{11}(k) \in L^2(\mathbb{R}^n)$. ▮

4.2. Proof of the Compactness of Family $\{\mu_t\}$

We now prove bound (3.4). Proposition 3.1 then can be deduced with the help of Prokhorov's Theorem, ref. 10, Lemma II.3.1, in a way similar to ref. 10, Thm. XII.5.2. Formulas (A.5), (A.2) and Proposition 4.1 imply

$$E(u(x,t)\, u(y,t)) =: q_t^{00}(x-y)$$

$$= \frac{1}{(2\pi)^n} \int e^{-ik(x-y)} \left[\frac{1+\cos 2|k|\, t}{2}\, \hat{q}_0^{00}(k) + \frac{\sin 2|k|\, t}{2|k|} (\hat{q}_0^{01}(k) + \hat{q}_0^{10}(k)) + \frac{1-\cos 2|k|\, t}{2|k|^2}\, \hat{q}_0^{11}(k) \right] dk, \tag{4.5}$$

where the integral converges and define a continuous function. Similar representations hold for all $i, j=0, 1$. Therefore, we have as in (1.5),

$$e_t := q_t^{11}(0) - \Delta q_t^{00}(0) + q_t^{00}(0) = \frac{1}{(2\pi)^n} \int (\hat{q}_t^{11}(k) + |k|^2\, \hat{q}_t^{00}(k) + \hat{q}_t^{00}(k))\, dk. \tag{4.6}$$

It remains to estimate the last integral. Equation (4.5) implies the following representation for $\hat{q}_t^{00}$,

$$\hat{q}_t^{00}(k) = \frac{1+\cos 2\,|k|\,t}{2}\, \hat{q}_0^{00}(k) + \frac{\sin 2\,|k|\,t}{2\,|k|}\, (\hat{q}_0^{01}(k) + \hat{q}_0^{10}(k)) + \frac{1-\cos 2\,|k|\,t}{2\,|k|^2}\, \hat{q}_0^{11}(k). \tag{4.7}$$

Similarly, formulas (A.5), (A.2) imply

$$\hat{q}_t^{11}(k) = |k|^2\, \frac{1-\cos 2\,|k|\,t}{2}\, \hat{q}_0^{00}(k) - |k|\, \frac{\sin 2\,|k|\,t}{2}\, (\hat{q}_0^{01}(k) + \hat{q}_0^{10}(k)) + \frac{1-\cos 2\,|k|\,t}{2}\, \hat{q}_0^{11}(k). \tag{4.8}$$

Therefore, (4.1) and (4.6) imply that $e_t \leqslant C_1(\varphi)\, e_0$. Hence, taking expectation in (2.4), we get (3.4):

$$E\, \|U_0(t)\, Y_0\|_R^2 = e_t\, |B_R| \leqslant C_1(\varphi)\, e_0\, |B_R|. \tag{4.9}$$

Here B_R denotes the ball $\{x \in \mathbb{R}^n : |x| \leqslant R\}$ and $|B_R|$ is its volume. ▮

Corollary 4.2. Bound (3.4) implies the convergence of the integrals in (2.8).

Bound (3.4) also implies, similarly to (2.13), that

$$\sup_{t \geqslant 0} \int |||Y|||_\delta^2\, \mu_t(dY) \leqslant C_\delta(\varphi)\, e_0 < \infty, \qquad \delta > 0. \tag{4.10}$$

This integral estimate implies the following corollary which we will use in Section 9.

Corollary 4.3. (i) Measures μ_t, $t \geqslant 0$, are concentrated in $\mathscr{H}_\delta$ for any $\delta > 0$, and the characteristic functionals $\hat{\mu}_t$ are equicontinuous in the dual Hilbert space $\mathscr{H}'_\delta$: for all $\Psi_1, \Psi_2 \in \mathscr{H}'_\delta$,

$$\begin{aligned}|\hat{\mu}_t(\Psi_1) - \hat{\mu}_t(\Psi_2)| &\leqslant \int |\exp(i\langle Y, \Psi_1 - \Psi_2\rangle) - 1| \; \mu_t(dY) \\ &\leqslant \int |\langle Y, \Psi_1 - \Psi_2\rangle| \; \mu_t(dY) \\ &\leqslant \int |||Y|||_\delta \cdot |||\Psi_1 - \Psi_2|||'_\delta \; \mu_t(dY) \\ &\leqslant C(\delta, \varphi, e_0) \; |||\Psi_1 - \Psi_2|||'_\delta, \qquad t \geqslant 0, \end{aligned} \tag{4.11}$$

where $|||\cdot|||'_\delta$ denotes the norm in $\mathscr{H}'_\delta$.

(ii) The quadratic forms $\mathscr{Q}_t(\Psi, \Psi)$ are equicontinuous in $\mathscr{H}'_\delta$: for all $\Psi_1, \Psi_2 \in \mathscr{H}'_\delta$,

$$\begin{aligned}&|\mathscr{Q}_t(\Psi_1, \Psi_1) - \mathscr{Q}_t(\Psi_2, \Psi_2)| \\ &\quad \leqslant C \, |\mathscr{Q}_t(\Psi_1 - \Psi_2, \Psi_1) + \mathscr{Q}_t(\Psi_2, \Psi_1 - \Psi_2)| \\ &\quad \leqslant C \int |\langle Y, \Psi_1 - \Psi_2\rangle| \, (|\langle Y, \Psi_1\rangle| + |\langle Y, \Psi_2\rangle|) \; \mu_t(dY) \\ &\quad \leqslant C \int |||Y|||^2_\delta \cdot |||\Psi_1 - \Psi_2|||'_\delta \cdot (|||\Psi_1|||'_\delta + |||\Psi_2|||'_\delta) \; \mu_t(dY) \\ &\quad \leqslant C(\delta, \varphi, e_0) \; |||\Psi_1 - \Psi_2|||'_\delta, \qquad t \geqslant 0. \end{aligned} \tag{4.12}$$

(iii) Therefore, the quadratic form $\mathscr{Q}_\infty(\Psi, \Psi)$ is continuous in $\mathscr{H}'_\delta$.

4.3. Convergence of the Covariance Functions

Here we prove the convergence of the CFs of measures μ_t. This convergence is used in Section 6.

Lemma 4.4. The following convergence holds as $t \to \infty$:

$$q_t^{ij}(z) \to q_\infty^{ij}(z), \qquad \forall z \in \mathbb{R}^n, \quad i, j = 0, 1. \tag{4.13}$$

Proof. (4.7) and (4.8) imply the convergence for $i = j$: the oscillatory terms there converge to zero as they are absolutely continuous and summable by Proposition 4.1. For $i \neq j$ the proof is similar. ∎

5. BERNSTEIN'S ARGUMENT FOR THE WAVE EQUATION

In this and the subsequent section we develop a version of Bernstein's "room-corridor" method. We use the standard integral representation for solutions, divide the domain of integration into "rooms" and "corridors" and evaluate their contribution. As a result, $\langle U_0(t)\, Y_0, \Psi \rangle$ is represented as the sum of weakly dependent random variables. We evaluate the variances of these random variables which will be important in next section.

First, we evaluate $\langle Y(t), \Psi \rangle$ in (3.5) by using the duality arguments. For $t \in \mathbb{R}$, introduce the operators $U'(t)$, $U'_0(t)$ in the Hilbert space $\mathscr{H}'_\delta$, which are adjoint to operators $U(t)$, $U_0(t)$ in $\mathscr{H}_\delta$. For example,

$$\langle Y, U'_0(t)\, \Psi \rangle = \langle U_0(t)\, Y, \Psi \rangle, \qquad \Psi \in \mathscr{D}, \quad Y \in \mathscr{H}_\delta, \quad t \in \mathbb{R}. \tag{5.1}$$

The adjoint groups admit a convenient description. Lemma 5.1 below displays that the action of groups $U'_0(t)$, $U'(t)$ coincides, respectively, with the action of $U_0(t)$, $U(t)$, up to the order of the components. In particular, $U'_0(t)$ is a continuous group in $\mathscr{D}$.

Lemma 5.1. For $\Psi = (\Psi^0, \Psi^1) \in \mathscr{D}$

$$U'_0(t)\, \Psi = (\dot{\phi}(\cdot, t), \phi(\cdot, t)), \qquad U'(t)\, \Psi = (\dot{\psi}(\cdot, t), \psi(\cdot, t)), \tag{5.2}$$

where $\phi(x, t)$ is the solution of Eq. (3.1) with the initial datum $(u_0, v_0) = (\Psi^1, \Psi^0)$ and $\psi(x, t)$ is the solution of Eq. (1.1) with the initial date $(u_0, v_0) = (\Psi^1, \Psi^0)$.

Proof. Differentiating (5.1) with $Y, \Psi \in \mathscr{D}$, we obtain

$$\langle Y, \dot{U}'_0(t)\, \Psi \rangle = \langle \dot{U}_0(t)\, Y, \Psi \rangle. \tag{5.3}$$

Group $U_0(t)$ has the generator (3.3) The generator of $U'_0(t)$ is the conjugate operator

$$\mathscr{A}'_0 = \begin{pmatrix} 0 & A_0 \\ 1 & 0 \end{pmatrix}. \tag{5.4}$$

Hence, Eq. (5.2) holds with $\ddot{\psi} = A_0 \psi$. For the group $U'(t)$ the proof is similar. ∎

Denote $\Phi(\cdot, t) = U'_0(t)\, \Psi$. Then (5.1) means that

$$\langle Y(t), \Psi \rangle = \langle Y_0, \Phi(\cdot, t) \rangle, \qquad t \in \mathbb{R}. \tag{5.5}$$

Remark. The representation (5.5) plays a central role in the proof of Proposition 3.2. A key observation is that $\Phi(x, t)$ is supported by an "inflated" cone of thickness $\approx \bar{r}$ where $\bar{r}$ is the diameter of supp Ψ. The last fact follows from the *strong* Huyghen's principle for group $U_0'(t)$ which holds for odd $n \geqslant 3$. Therefore, the scalar product $\langle Y_0, \Phi(\cdot, t)\rangle$ is represented as an integral over the "spherical slab" of width $\approx \bar{r}$. This replaces, for a general $n \geqslant 3$, the Kirchhoff integral (1.10) written for $n = 3$.

Next we introduce "room-corridor" partition of the space $\mathbb{R}^n$. Given $t > 0$, choose $d \equiv d_t \geqslant 1$ and $\rho \equiv \rho_t > 0$. Asymptotical relations between t, d_t and ρ_t are specified below. Define $h = d + \rho$ and

$$a^j = jh, \qquad b^j = a^j + d, \qquad j \in \mathbb{Z}. \tag{5.6}$$

We call the slabs $R_t^j = \{x \in \mathbb{R}^n : a^j \leqslant x^n \leqslant b^j\}$ "rooms" and $C_t^j = \{x \in \mathbb{R}^n : b^j \leqslant x^n \leqslant a_{j+1}\}$ "corridors." Here $x = (x^1,..., x^n)$, d is the width of a room, and ρ of a corridor.

Denote by χ_r the indicator of the interval $[0, d]$ and χ_c that of $[d, h]$ so that $\sum_{j \in \mathbb{Z}} (\chi_r(s - jh) + \chi_c(s - jh)) = 1$ for (almost all) $s \in \mathbb{R}$. The following decomposition holds:

$$\langle Y_0, \Phi(\cdot, t)\rangle = \sum_{j \in \mathbb{Z}} (\langle Y_0, \chi_r^j \Phi(\cdot, t)\rangle + \langle Y_0, \chi_c^j \Phi(\cdot, t)\rangle), \tag{5.7}$$

where $\chi_r^j := \chi_r(x^n - jh)$ and $\chi_c^j := \chi_c(x^n - jh)$. Consider random variables r_t^j, c_t^j, where

$$r_t^j = \langle Y_0, \chi_r^j \Phi(\cdot, t)\rangle, \qquad c_t^j = \langle Y_0, \chi_c^j \Phi(\cdot, t)\rangle, \qquad j \in \mathbb{Z}. \tag{5.8}$$

Then (5.7) and (5.5) imply

$$\langle U_0(t)\, Y_0, \Psi\rangle = \sum_{j \in \mathbb{Z}} (r_t^j + c_t^j). \tag{5.9}$$

The series in (5.9) is actually a finite sum. In fact, (5.4) and (A.1) imply that in the Fourier representation, $\dot{\hat{\Phi}}(k, t) = \hat{\mathscr{A}}_0'(k)\, \hat{\Phi}(k, t)$ and $\hat{\Phi}(k, t) = \hat{\mathscr{G}}_t'(k)\, \hat{\Psi}(k)$. Therefore,

$$\Phi(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-ikx} \hat{\mathscr{G}}_t'(k)\, \hat{\Psi}(k)\, dk. \tag{5.10}$$

This can be rewritten as a convolution

$$\Phi(\cdot, t) = \mathscr{R}_t * \Psi, \tag{5.11}$$

where $\mathscr{R}_t = F^{-1}\hat{\mathscr{G}}'_t$. The support supp $\Psi \subset B_{\bar{r}}$ with an $\bar{r} > 0$. Then the convolution representation (5.11) implies that the support of the function Φ at $t > 0$ is a subset of an "inflated light cone"

$$\text{supp}\, \Phi(x, t) \subset \{(x, t) \in \mathbb{R}^n \times \mathbb{R}_+ : -\bar{r} \leqslant |x| - t \leqslant \bar{r}\}. \tag{5.12}$$

as $\mathscr{R}_t(x)$ is supported by the light cone $|x| = t$ as n is odd $\geqslant 3$. The last fact follows from the general Herglotz–Petrovskii formulas (see, e.g., ref. 18, (II.4.4.11)) and is known as *strong* Huyghen's principle. Finally, (5.8) implies

$$r_t^j = c_t^j = 0 \qquad \text{for} \quad jh + t < -\bar{r} \qquad \text{or} \quad jh - t > \bar{r}. \tag{5.13}$$

Therefore, series (5.9) becomes a sum

$$\langle U_0(t)\, Y_0, \Psi \rangle = \sum_{-N_t}^{N_t} (r_t^j + c_t^j), \qquad N_t \sim \frac{t}{h}, \tag{5.14}$$

as $h \geqslant 1$.

Lemma 5.2. Let $n \geqslant 1$, $m > 0$, and S0–S3 hold. The following bounds hold for $t > 1$ and $\forall j$:

$$E\, |r_t^j|^2 \leqslant C(\Psi)\, d_t/t, \qquad E\, |c_t^j|^2 \leqslant C(\Psi)\, \rho_t/t. \tag{5.15}$$

Proof. We discuss the first bound in (5.15) only, the second is done in a similar way.

Step 1. Rewrite the left hand side as the integral of covariance matrices. Definition (5.8) and Corollary 2.14 imply by Fubini's Theorem that

$$E\, |r_t^j|^2 = \langle \chi_r^j(x^n)\, \chi_r^j(y^n)\, q_0(x - y), \Phi(x, t) \otimes \Phi(y, t) \rangle. \tag{5.16}$$

The following bound holds true (cf. ref. 19, Thm. XI.19(c)):

$$\sup_{x \in \mathbb{R}^n} |\Phi(x, t)| = \mathcal{O}(t^{-\frac{n-1}{2}}), \qquad t \to \infty. \tag{5.17}$$

In fact, (5.10) and (A.2) imply that Φ can be written as the sum

$$\Phi(x, t) = \frac{1}{(2\pi)^n} \sum_{\pm} \int_{\mathbb{R}^n} e^{-i(kx \mp |k|\, t)} a^{\pm}(|k|)\, \hat{\Psi}(k)\, dk, \tag{5.18}$$

where $a^{\pm}(|k|)$ is a matrix whose entries are linear functions in $|k|$ or $1/|k|$. Let us prove the asymptotics (5.17) along each ray $x = vt + x_0$ with $|v| = 1$, then it holds uniformly in $x \in \mathbb{R}^n$ owing to (5.12). In polar coordinates, we get from (5.18),

$$\Phi(vt+x_0, t) = \frac{1}{(2\pi)^n} \sum_{\pm} \int_0^{\infty} \left(\int_{|k|=r} e^{-i(kv \mp |k|)\, t - ikx_0} a^{\pm}(|k|)\, \hat{\Psi}(k)\, dS(k) \right) dr. \tag{5.19}$$

This is a sum of oscillatory integrals with the phase functions $\phi_{\pm}(k) = kv \pm |k|$. The standard form of the method of stationary phase is not applicable here as the set of stationary points $\{k \in \mathbb{R}^n : \nabla \phi_{\pm}(k) = 0\}$, is a ray $v = \pm k/|k|$, and the Hessian is degenerate everywhere. On the other hand, restricted to the sphere $|k| = r$ with a fixed $r > 0$, each phase function $\phi^r_{\pm} := \phi_{\pm}|_{|k|=r} = kv \pm r$, has two stationary points $\pm vr$, and the Hessian is nondegenerate everywhere:

$$\operatorname{rank}(\operatorname{Hess} \phi^r_{\pm}(k)) = n-1, \qquad |k| = r. \tag{5.20}$$

Hence, the inner integral in (5.19) is $\mathcal{O}(t^{-(n-1)/2})$ according to the standard method of stationary phase.[20] At last, for the integral in r in (5.19), has the same asymptotics in t as $\hat{\Psi}(k)$ decay rapidly at infinity.

Step 2. According to (5.12) and (5.17), Eq. (5.16) implies that 12

$$E\, |r^j_t|^2 \leqslant C t^{-n+1} \int_{S^{\bar{r}}_t \times S^{\bar{r}}_t} \chi^j_r(x^n)\, \|q_0(x-y)\|\, dx\, dy, \tag{5.21}$$

where $S^{\bar{r}}_t$ is an "inflated sphere" $\{x \in \mathbb{R}^n : -\bar{r} \leqslant |x| - t \leqslant \bar{r}\}$ and $\|q_0(x-y)\|$ stands for the norm of the 2×2-matrix, $(q^{ij}_0(x-y))$. The estimate (2.16) implies then

$$E\, |r^j_t|^2 \leqslant C t^{-n+1} \int_{S^{\bar{r}}_t \times S^{\bar{r}}_t} \chi^j_r(x^n)\, \varphi^{1/2}(|x-y|)\, dx\, dy, \tag{5.22}$$

For large t, this integral can be reduced to the product of the spheres $S_t = \{x \in \mathbb{R}^n : |x| = t\}$:

$$E\, |r^j_t|^2 \leqslant C_1 \bar{r}^2 t^{-n+1} \int_{S_t} \chi^j_r(x^n) \left(\int_{S_t} \varphi^{1/2}(|x-y|)\, dS(y) \right) dS(x), \tag{5.23}$$

where dS is a Lebesgue measure on the sphere. The inner integral can be estimated by a direct computation owing to (2.16):

$$\int_{S_t} \varphi^{1/2}(|x-y|)\, dS(y) = C(n) \int_0^{2t} r^{n-1} \varphi^{1/2}(r)\, dr \leqslant C(n)\, \bar{\varphi}, \qquad t \geqslant 0. \tag{5.24}$$

Therefore, (5.23) and (2.12) imply

$$E\, |r_t^j|^2 \leqslant C_1(n)\, \bar{r}^2 t^{-n+1} \int_{S_t} \chi_r^j(x^n)\, dS(x) \leqslant C_2(n)\, d_t/t. \quad \blacksquare$$

6. CONVERGENCE OF THE CHARACTERISTIC FUNCTIONALS

In this section we complete the proof of Proposition 3.2. As was said, we use a version of the CLT developed by Ibragimov and Linnik. This gives the convergence to an equilibrium Gaussian measure. If $\mathcal{Q}_\infty(\Psi, \Psi) = 0$, Proposition 3.2 is obvious. Thus, we may assume that

$$\mathcal{Q}_\infty(\Psi, \Psi) \neq 0. \tag{6.1}$$

Choose $0 < \delta < 1$ and

$$\rho_t \sim t^{1-\delta}, \qquad d_t \sim \frac{t}{\ln t}, \qquad t \to \infty. \tag{6.2}$$

Lemma 6.1. The following limit holds true:

$$N_t \left(\varphi(\rho_t) + \left(\frac{\rho_t}{t} \right)^{1/2} \right) + N_t^2 \left(\varphi^{1/2}(\rho_t) + \frac{\rho_t}{t} \right) \to 0, \qquad t \to \infty. \tag{6.3}$$

Proof. Equation (6.3) follows from (4.4) as (6.2) and (5.14) imply that $N_t \sim \ln t$. $\blacksquare$

By the triangle inequality,

$$\begin{aligned} |\hat{\mu}_t(\Psi) - \hat{\mu}_\infty(\Psi)| &\leqslant |E \exp\{i \langle U_0(t)\, Y_0, \Psi \rangle\} - E \exp\{i \textstyle\sum_t r_t^j\}| \\ &\quad + |\exp\{-\tfrac{1}{2} \textstyle\sum_t E\, |r_t^j|^2\} - \exp\{-\tfrac{1}{2}\, \mathcal{Q}_\infty(\Psi, \Psi)\}| \\ &\quad + |E \exp\{i \textstyle\sum_t r_t^j\} - \exp\{-\tfrac{1}{2} \textstyle\sum_t E\, |r_t^j|^2\}| \\ &\equiv I_1 + I_2 + I_3, \end{aligned} \tag{6.4}$$

where the sum $\sum_t$ stands for $\sum_{j=-N_t}^{N_t}$. We are going to show that all summands I_1, I_2, I_3 tend to zero as $t \to \infty$.

Step (i). Equation (5.14) implies

$$I_1 = |E \exp\{i \textstyle\sum_t r_t^j\}(\exp\{i \sum_t c_t^j\} - 1)| \leqslant \sum_t E\,|c_t^j| \leqslant \sum_t (E\,|c_t^j|^2)^{1/2}. \qquad (6.5)$$

From (6.5), (5.15) and (6.3) we obtain that

$$I_1 \leqslant CN_t(\rho_t/t)^{1/2} \to 0, \qquad t \to \infty. \qquad (6.6)$$

Step (ii). By the triangle inequality,

$$\begin{aligned} I_2 &\leqslant \tfrac{1}{2}\,|\textstyle\sum_t E\,|r_t^j|^2 - \mathcal{Q}_\infty(\Psi, \Psi)| \leqslant \tfrac{1}{2}\,|\mathcal{Q}_t(\Psi, \Psi) - \mathcal{Q}_\infty(\Psi, \Psi)| \\ &\quad + \tfrac{1}{2}\,|E(\textstyle\sum_t r_t^j)^2 - \sum_t E\,|r_t^j|^2| + \tfrac{1}{2}\,|E(\sum_t r_t^j)^2 - \mathcal{Q}_t(\Psi, \Psi)| \\ &\equiv I_{21} + I_{22} + I_{23}, \end{aligned} \qquad (6.7)$$

where $\mathcal{Q}_t$ is a quadratic form with the integral kernel $(Q_t^{ij}(x, y))$. Equation (4.13) implies that $I_{21} \to 0$. As to I_{22}, we first have that

$$I_{22} \leqslant \sum_{j<l} E\,|r_t^j r_t^l|. \qquad (6.8)$$

The next lemma is a corollary of ref. 16, Lemma 17.2.3.

Lemma 6.2. Let ξ be a complex random value measurable with respect to σ-algebra $\sigma(\mathcal{A})$, η with respect to σ-algebra $\sigma(\mathcal{B})$, and $\operatorname{dist}(\mathcal{A}, \mathcal{B}) \geqslant r > 0$.

(i) Let $(E\,|\xi|^2)^{1/2} \leqslant a$, $(E\,|\eta|^2)^{1/2} \leqslant b$. Then

$$|E\xi\eta - E\xi E\eta| \leqslant Cab\varphi^{1/2}(r).$$

(ii) Let $|\xi| \leqslant a$, $|\eta| \leqslant b$ almost sure. Then

$$|E\xi\eta - E\xi E\eta| \leqslant Cab\varphi(r).$$

We apply Lemma 6.2 to deduce that $I_{22} \to 0$ as $t \to \infty$. Note that $r_t^j = \langle Y_0(x), \chi_r^j(x^n)(\mathcal{R}_t * \Psi)\rangle$ is measurable with respect to the σ-algebra $\sigma(R_t^j)$. The distance between the different rooms R_t^j is greater or equal to ρ_t

according to (5.6). Then (6.8) and S1, S3 imply, together with Lemma 6.2(i), that

$$I_{22} \leqslant CN_t^2 \varphi^{1/2}(\rho_t), \tag{6.9}$$

which goes to 0 as $t \to \infty$ because of (5.15) and (6.3). Finally, it remains to check that $I_{23} \to 0$, $t \to \infty$. By the Cauchy–Schwartz inequality,

$$\begin{aligned} I_{23} &\leqslant |E(\textstyle\sum_t r_t^j)^2 - E(\sum_t r_t^j + \sum_t c_t^j)^2| \\ &\leqslant CN_t \textstyle\sum_t E\,|c_t^j|^2 + C(E(\sum_t r_t^j)^2)^{1/2}\,(N_t \sum_t E\,|c_t^j|^2)^{1/2}. \end{aligned} \tag{6.10}$$

Then (5.15), (6.8) and (6.9) imply

$$E(\textstyle\sum_t r_t^j)^2 \leqslant \sum_t E\,|r_t^j|^2 + 2\sum_{j<l} E\,|r_t^j r_t^l| \leqslant CN_t d_t/t + C_1 N_t \varphi^{1/2}(\rho_t) \leqslant C_2 < \infty.$$

Now (5.15), (6.10) and (6.3) yields

$$I_{23} \leqslant C_1 N_t^2 \rho_t/t + C_2 N_t (\rho_t/t)^{1/2} \to 0, \qquad t \to \infty. \tag{6.11}$$

So, all terms I_{21}, I_{22}, I_{23} in (6.7) tend to zero. Then (6.7) implies that

$$I_2 \leqslant \tfrac{1}{2}\,|\textstyle\sum_t E\,|r_t^j|^2 - \mathcal{Q}_\infty(\Psi, \Psi)| \to 0, \qquad t \to \infty. \tag{6.12}$$

Step (iii). It remains to verify that

$$I_3 = |E \exp\{i \textstyle\sum_t r_t^j\} - \exp\{-\tfrac{1}{2} E(\sum_t r_t^j)^2\}| \to 0, \qquad t \to \infty. \tag{6.13}$$

Using Lemma 6.2(ii) we obtain:

$$\begin{aligned} &\left| E \exp\left\{ i \sum_t r_t^j \right\} - \prod_{-N_t}^{N_t} E \exp\{ir_t^j\} \right| \\ &\quad\leqslant \left| E \exp\{ir_t^{-N_t}\} \exp\left\{ i \sum_{-N_t+1}^{N_t} r_t^j \right\} - E \exp\{ir_t^{-N_t}\}\, E \exp\left\{ i \sum_{-N_t+1}^{N_t} r_t^j \right\} \right| \\ &\quad\quad + \left| E \exp\{ir_t^{-N_t}\}\, E \exp\left\{ i \sum_{-N_t+1}^{N_t} r_t^j \right\} - \prod_{-N_t}^{N_t} E \exp\{ir_t^j\} \right| \\ &\quad\leqslant C\varphi(\rho_t) + \left| E \exp\left\{ i \sum_{-N_t+1}^{N_t} r_t^j \right\} - \prod_{-N_t+1}^{N_t} E \exp\{ir_t^j\} \right|. \end{aligned}$$

We then apply Lemma 6.2(ii) recursively and get, according to Lemma 6.1,

$$\left| E \exp\left\{ i \sum_t r_t^j \right\} - \prod_{-N_t}^{N_t} E \exp\{ i r_t^j \} \right| \leqslant C N_t \varphi(\rho_t) \to 0, \qquad t \to \infty. \tag{6.14}$$

It remains to check that

$$\left| \prod_{-N_t}^{N_t} E \exp\{ i r_t^j \} - \exp\left\{ -\tfrac{1}{2} \sum_t E\, |r_t^j|^2 \right\} \right| \to 0, \qquad t \to \infty. \tag{6.15}$$

According to the standard statement of the CLT (see, e.g., ref. 21, Thm. 4.7), it suffices to verify the Lindeberg condition: $\forall \varepsilon > 0$

$$\frac{1}{\sigma_t} \sum_t E_{\varepsilon \sqrt{\sigma_t}}\, |r_t^j|^2 \to 0, \qquad t \to \infty. \tag{6.16}$$

Here $\sigma_t \equiv \sum_t E\, |r_t^j|^2$, and $E_\delta f \equiv E X_\delta f$, where X_δ is the indicator of the event $|f| > \delta^2$. Note that (6.12) and (6.1) imply that

$$\sigma_t \to \mathscr{Q}_\infty(\Psi, \Psi) \neq 0, \qquad t \to \infty.$$

Hence it remains to verify that $\forall \varepsilon > 0$

$$\sum_t E_\varepsilon\, |r_t^j|^2 \to 0, \qquad t \to \infty. \tag{6.17}$$

We check (6.17) in Section 7. This will complete the proof of Proposition 3.2. ▮

7. THE LINDEBERG CONDITION

The proof of (6.17) can be reduced to the case when for some $\Lambda \geqslant 0$ we have, almost sure that

$$|u_0(x)| + |v_0(x)| \leqslant \Lambda < \infty, \qquad x \in \mathbb{R}^n. \tag{7.1}$$

Then the proof of (6.17) is reduced to the convergence

$$\sum_t E\, |r_t^j|^4 \to 0, \qquad t \to \infty \tag{7.2}$$

by using Chebyshev's inequality. The general case can be covered by standard cutoff arguments taking into account that bound (5.15) for $E\, |r_t^j|^2$

depends only on e_0 and φ. The last fact is evident from (5.21)–(5.24). We deduce (7.2) from

Theorem 7.1. Let the conditions of Theorem B hold and assume that (7.1) is fulfilled. Then for any $\Psi \in \mathscr{D}$ there exists a constant $C(\Psi)$ such that

$$E\,|r_t^j|^4 \leqslant C(\Psi)\,\Lambda^4 d_t^2/t^2, \qquad t > 1. \tag{7.3}$$

Step 1. Given four points $x_1, x_2, x_3, x_4 \in \mathbb{R}^n$, set:

$$M_0^{(4)}(x_1,..., x_4) = E(Y_0(x_1) \otimes \cdots \otimes Y_0(x_4)).$$

Then, similarly to (5.16), Eqs. (7.1) and (5.8) imply by the Fubini Theorem that

$$E\,|r_t^j|^4 = \langle \chi_r^j(x_1^n) \cdots \chi_r^j(x_4^n)\, M_0^{(4)}(x_1,..., x_4), \Phi(x_1, t) \otimes \cdots \otimes \Phi(x_4, t)\rangle. \tag{7.4}$$

Let us analyse the domain of the integration $(\mathbb{R}^n)^4$ in the RHS of (7.4). We partition $(\mathbb{R}^n)^4$ into three parts W_2, W_3 and W_4:

$$(\mathbb{R}^n)^4 = \bigcup_{i=2}^{4} W_i, \quad W_i = \{\bar{x} = (x_1, x_2, x_3, x_4) \in (\mathbb{R}^n)^4 : |x_1 - x_i| = \max_{p=2,3,4} |x_1 - x_p|\}. \tag{7.5}$$

Furthermore, given $\bar{x} = (x_1, x_2, x_3, x_4) \in W_i$, divide $\mathbb{R}^n$ into three parts S_j, $j = 1, 2, 3$: $\mathbb{R}^n = S_1 \cup S_2 \cup S_3$, by two hyperplanes orthogonal to the segment $[x_1, x_i]$ and partitioning it into three equal segments, where $x_1 \in S_1$ and $x_i \in S_3$. Denote by x_p, x_q the two remaining points with $p, q \neq 1, i$. Set: $\mathscr{A}_i = \{\bar{x} \in W_i : x_p \in S_1, x_q \in S_3\}$, $\mathscr{B}_i = \{\bar{x} \in W_i : x_p, x_q \notin S_1\}$ and $\mathscr{C}_i = \{\bar{x} \in W_i : x_p, x_q \notin S_3\}$, $i = 2, 3, 4$. Then $W_i = \mathscr{A}_i \cup \mathscr{B}_i \cup \mathscr{C}_i$. Define the function $\mathrm{m}_0^{(4)}(\bar{x})$, $\bar{x} \in (\mathbb{R}^n)^4$, in the following way:

$$\mathrm{m}_0^{(4)}(\bar{x})|_{W_i} = \begin{cases} M_0^{(4)}(\bar{x}) - q_0(x_1 - x_p) \otimes q_0(x_i - x_q), & \bar{x} \in \mathscr{A}_i, \\ M_0^{(4)}(\bar{x}), & \bar{x} \in \mathscr{B}_i \cup \mathscr{C}_i. \end{cases} \tag{7.6}$$

This determines $\mathrm{m}_0^{(4)}(\bar{x})$ correctly for almost all quadruples $\bar{x}$. Note that

$$\begin{aligned}
&\langle \chi_r^j(x_1^n) \cdots \chi_r^j(x_4^n)\, q_0(x_1 - x_p) \otimes q_0(x_i - x_q), \Phi(x_1, t) \otimes \cdots \otimes \Phi(x_4, t)\rangle \\
&\quad = \langle \chi_r^j(x_1^n)\, \chi_r^j(x_p^n)\, q_0(x_1 - x_p), \Phi(x_1, t) \otimes \Phi(x_p, t)\rangle \\
&\qquad \times \langle \chi_r^j(x_i^n)\, \chi_r^j(x_q^n)\, q_0(x_i - x_q), \Phi(x_i, t) \otimes \Phi(x_q, t)\rangle.
\end{aligned}$$

Each factor here is bounded by $C(\Psi)\, d_t/t$. Similarly to (5.15), this can be deduced from an expression of type (5.16) for the factors. Therefore, the proof of (7.3) reduces to the proof of the bound

$$I_t := |\langle \chi_r^j(x_1^n) \cdots \chi_r^j(x_4^n)\, \mathrm{m}_0^{(4)}(x_1,..., x_4), \Phi(x_1, t) \otimes \cdots \otimes \Phi(x_4, t)\rangle| \leqslant C(\Psi)\, \Lambda^4 d_t^2/t^2, \qquad t > 1. \tag{7.7}$$

Step 2. Similarly to (5.21), Eq. (5.17) implies,

$$I_t \leqslant C(\Psi)\, t^{-2n+2} \int_{(S_t^{\bar r})^4} \chi_r^j(x_1^n) \cdots \chi_r^j(x_4^n)\, |\mathrm{m}_0^{(4)}(x_1,..., x_4)|\, dx_1\, dx_2\, dx_3\, dx_4, \tag{7.8}$$

where $S_t^{\bar r}$ is an "inflated sphere" $\{x \in \mathbb{R}^n : -\bar r \leqslant |x| - t \leqslant \bar r\}$. Let us estimate $\mathrm{m}_0^{(4)}$ using Lemma 6.2(ii).

Lemma 7.2. For each $i = 2, 3, 4$ and almost all $\bar x \in W_i$ the following bound holds

$$|\mathrm{m}_0^{(4)}(x_1,..., x_4)| \leqslant C\Lambda^4 \varphi(|x_1 - x_i|/3). \tag{7.9}$$

Proof. For $\bar x \in \mathscr{A}_i$ we apply Lemma 6.2(ii) to $\mathbb{R}^2 \otimes \mathbb{R}^2$-valued random variables $\xi = Y_0(x_1) \otimes Y_0(x_p)$ and $\eta = Y_0(x_i) \otimes Y_0(x_q)$. Then (7.1) implies the bound for almost all $\bar x \in \mathscr{A}_i$

$$|\mathrm{m}_0^{(4)}(\bar x)| \leqslant C\Lambda^4 \varphi(|x_1 - x_i|/3). \tag{7.10}$$

For $\bar x \in \mathscr{B}_i$, we apply Lemma 6.2(ii) to $\xi = Y_0(x_1)$ and $\eta = Y_0(x_p) \otimes Y_0(x_q) \otimes Y_0(x_i)$. Then S0 implies a similar bound for almost all $\bar x \in \mathscr{B}_i$,

$$|\mathrm{m}_0^{(4)}(\bar x)| = |M_0^{(4)}(\bar x) - EY_0(x_1) \otimes E(Y_0(x_p) \otimes Y_0(x_q) \otimes Y_0(x_i))| \leqslant C\Lambda^4 \varphi(|x_1 - x_i|/3), \tag{7.11}$$

and the same for almost all $\bar x \in \mathscr{C}_i$. ▮

Step 3. It remains to prove the following bounds for each $i = 2, 3, 4$ (cf. (5.22)):

$$V_i(t) := \int_{(S_t^{\bar r})^4} \chi_r^j(x_1^n) \cdots \chi_r^j(x_4^n)\, X_i(\bar x)\, \varphi(|x_1 - x_i|/3)\, dx_1\, dx_2\, dx_3\, dx_4 \leqslant C d_t^2 t^{2n-4}, \tag{7.12}$$

where X_i is an indicator of the set W_i. In fact, this integral does not depend on i, hence set $i=2$ in the integrand. Similarly to (5.23), this integral can be reduced to the product of four spheres $S_t = S_t^0$: for large t,

$$V_i(t) \leqslant C\bar{r}^4 \int_{(S_t)^2} \chi_r^j(x_1^n)\, \varphi(|x_1-x_2|/3)$$
$$\times \left[\int_{S_t} \chi_r^j(x_3^n) \left(\int_{S_t} X_2(\bar{x})\, dS(x_4) \right) dS(x_3) \right] dS(x_1)\, dS(x_2). \qquad (7.13)$$

Now a key observation is that the inner integral in $dS(x_4)$ is $\mathcal{O}(|x_1-x_2|^{n-1})$ as $X_2(\bar{x})=0$ for $|x_4-x_1|>|x_1-x_2|$. This implies

$$V_i(t) \leqslant C\bar{r}^4 \int_{S_t} \chi_r^j(x_1^n) \left(\int_{S_t} \varphi(|x_1-x_2|/3)\, |x_1-x_2|^{n-1}\, dS(x_2) \right) dS(x_1)$$
$$\times \int_{S_t} \chi_r^j(x_3^n)\, dS(x_3). \qquad (7.14)$$

The inner integral in $dS(x_2)$ can be estimated by a direct computation: similarly to (5.24),

$$\int_{S_t} \varphi(|x_1-x_2|/3)\, |x_1-x_2|^{n-1}\, dS(x_2)$$
$$= C(n) \int_0^{2t} r^{2n-3} \varphi(r/3)\, dr$$
$$\leqslant C_1(n) \sup_{r\in[0,2t]} r^{n-1}\varphi^{1/2}(r/3) \int_0^{2t} r^{n-2}\varphi^{1/2}(r/3)\, dr. \qquad (7.15)$$

The "sup" and the last integral are bounded by (4.4) and (2.12), respectively. Therefore, (7.12) follows from (7.14). This completes the proof of Theorem 7.1. ∎

Proof of Convergence (7.2). The estimate (7.3) implies, since $d_t \leqslant h \sim t/N_t$,

$$\sum_t E\,|r_t^j|^4 \leqslant \frac{C\,\Lambda^4 d_t^2}{t^2}\, N_t \leqslant \frac{C_1 \Lambda^4}{N_t} \to 0, \qquad N_t \to \infty. \quad ∎$$

8. SCATTERING THEORY FOR INFINITE ENERGY SOLUTIONS

As was said in Sections 1 and 2, we reduce the proof of Theorem A to Theorem B by using a special version of the scattering theory, for solutions of infinite energy. Recall, $U(t)$, $U_0(t)$ are the dynamical groups of the Cauchy problems (1.2), (3.2), respectively.

Theorem 8.1. Let E1–E3 hold, and $n \geqslant 3$ be odd. Then there exist $\delta, \gamma > 0$ and linear continuous operators Θ, $\rho(t)$: $\mathscr{H}_\delta \to \mathscr{H}$ such that for $Y_0 \in \mathscr{H}_\delta$

$$U(t)\, Y_0 = \Theta U_0(t)\, Y_0 + \rho(t)\, Y_0, \qquad t \geqslant 0, \tag{8.1}$$

and for any $R > 0$ there exists a constant $C = C(R, \delta, \gamma)$, such that for $Y_0 \in \mathscr{H}_\delta$

$$\|\rho(t)\, Y_0\|_R \leqslant C\, e^{-\gamma t}\, |||Y_0|||_\delta, \qquad t \geqslant 0. \tag{8.2}$$

We deduce Theorem 8.1 with the help of duality from a special version of the finite energy scattering theory that is developed below. Denote $|||\cdot|||'_\delta$ the norm in the Hilbert space $\mathscr{H}'_\delta$, dual to $\mathscr{H}_\delta$.

Lemma 8.2. The following bound holds true:

$$|||U'_0(t)\, \Psi|||'_\delta \leqslant C e^{\delta\, |t|}\, |||\Psi|||'_\delta, \qquad \forall \Psi \in \mathscr{H}'_\delta, \quad t \in \mathbb{R}. \tag{8.3}$$

Lemma 8.2 follows by duality from Lemma 8.3:

Lemma 8.3. Let E1–E2 hold. Then $\forall \delta > 0$ the operator $U_0(t)$ is continuous in $\mathscr{H}_\delta$, and there exists a constant $C = C(\delta) > 0$ such that for $Y_0 \in \mathscr{H}_\delta$

$$|||U_0(t)\, Y_0|||_\delta \leqslant C e^{\delta\, |t|}\, |||Y_0|||_\delta, \qquad t \in \mathbb{R}. \tag{8.4}$$

Proof. It suffices to consider $Y_0 \in \mathscr{D}$. Then $U_0(t)\, Y_0 \in \mathscr{D}$. Denote

$$E_\delta(Y) \equiv \int e^{-2\delta\, |x|} (|v(x)|^2 + |\nabla u(x)|^2)\, dx, \qquad Y = (u, v) \in \mathscr{H}_\delta.$$

Then the derivative

$$\dot{E}_\delta(U_0(t)\, Y_0) = 2 \int e^{-2\delta\, |x|} (\dot{u}(x, t)\, \ddot{u}(x, t) + \nabla u(x, t)\, \nabla \dot{u}(x, t))\, dx.$$

Substituting $\ddot{u}(x, t) = \Delta u(x, t)$ and integrating by parts, we obtain

$$\dot{E}_\delta(U_0(t)\, Y_0) = -2 \int \nabla e^{-2\delta\, |x|} \cdot \nabla u(x, t)\, \dot{u}(x, t)\, dx.$$

Then $|\dot{E}_\delta(U_0(t)\,Y_0)| \leqslant 2\delta E_\delta(U_0(t)\,Y_0)$ by the Young inequality. Therefore, the Gronwall inequality implies

$$E_\delta(U_0(t)\,Y_0) \leqslant e^{2\delta\,|t|} E_\delta(Y_0).$$

In other words,

$$\int e^{-2\delta\,|x|}(|\dot{u}(x,t)|^2 + |\nabla u(x,t)|^2)\,dx \leqslant e^{2\delta\,|t|} E_\delta(Y_0). \tag{8.5}$$

It remains to estimate the norm $|||u(\cdot,t)|||_\delta$, where

$$|||u(\cdot,t)|||_\delta^2 \equiv \int \exp(-2\delta\,|x|)\,|u(x,t)|^2\,dx.$$

In fact, we have:

$$|||u(\cdot,t)|||_\delta \leqslant |||u^0(x)|||_\delta + \left|\left|\left|\int_0^t \dot{u}(\cdot,\bar{r})\,d\bar{r}\right|\right|\right|_\delta \leqslant |||u^0(x)|||_\delta + \left|\int_0^t |||\dot{u}(\cdot,\bar{r})|||_\delta\,d\bar{r}\right|.$$

Using (8.5), we get

$$|||u(\cdot,t)|||_\delta \leqslant Ce^{\delta\,|t|}\,|||Y_0|||_\delta. \quad \blacksquare$$

Now we employ Vainberg's bounds for the local energy decay; this plays the key role in the proof of Theorem 8.1. We use the Sobolev space $\mathscr{H}_{(R)} = H^1(B_R) \oplus L^2(B_R)$ with the norm $\|\cdot\|_{(R)}$, $R > 0$. Recall that $\overset{0}{H}{}^{-1}(B_R)$ is a completion of $D_R = \{\psi \in D : \operatorname{supp}\psi \subset B_R\}$ in the Hilbert norm of the Sobolev space $H^{-1}(\mathbb{R}^n)$. We will use the following convenient description of a dual space (see, e.g., ref. 22, Section I.3.4).

Lemma 8.4. $\overset{0}{H}{}^{-1}(B_R)$ is the dual to the Hilbert space $H^1(B_R)$ with respect to the scalar product $\langle\cdot,\cdot\rangle$.

Corollary 8.5. The dual space to the Hilbert space $\mathscr{H}_{(R)}$ with respect to $\langle\cdot,\cdot\rangle$ is

$$\mathscr{H}'_{(R)} = \overset{0}{H}{}^{-1}(B_R) \oplus L^2(B_R). \tag{8.6}$$

Note that $\mathscr{H}'_{(R)}$ is a subspace of $\mathscr{H}'_\delta$ with any $\delta \in \mathbb{R}$.

Definition 8.6. $\mathscr{H}'$ denotes the space $\bigcup_{R>0} \mathscr{H}'_{(R)}$ endowed with the following convergence: a sequence Ψ_n converges to Ψ in $\mathscr{H}'$ iff $\exists R > 0$ s.t. all $\Psi_n \in \mathscr{H}'_{(R)}$, and Ψ_n converge to Ψ in the norm of $\mathscr{H}'_{(R)}$.

Below, we consider the continuity of the maps from $\mathscr{H}'$ only in the sense of the sequential continuity. Vainberg's results imply the following lemma which we prove in Appendix A.

Lemma 8.7. Let E1–E3 hold, and let $n \geqslant 3$ be odd. Then $\forall R, R_0 > 0$ there exist constants $\alpha, C(R, R_0) > 0$ and $T = T(R, R_0) > 0$ such that for $\Psi \in \mathscr{H}'_{(R)}$

$$\|U'(t)\,\Psi\|_{L^2(B_{R_0}) \oplus H^1(B_{R_0})} \leqslant C(R, R_0)\, e^{-\alpha t}\, \|\Psi\|'_{(R)}, \qquad t \geqslant T, \tag{8.7}$$

where $\|\cdot\|'_{(R)}$ denotes the norm of the dual Hilbert space $\mathscr{H}'_{(R)}$.

Now we are in position to discuss our version of the scattering theory for solutions of finite energy.

Proposition 8.8. Let E1–E3 hold, and $n \geqslant 3$ be odd. Then there exist $\delta, \gamma > 0$ and linear continuous operators $W, r(t)$: $\mathscr{H}' \to \mathscr{H}'_\delta$ such that for $\Psi \in \mathscr{H}'$

$$U'(t)\,\Psi = U'_0(t)\, W\Psi + r(t)\,\Psi, \qquad t \geqslant 0, \tag{8.8}$$

and for any $R > 0$ there exists a constant $C_R = C(R, \delta, \gamma)$ such that

$$\|r(t)\,\Psi\|'_\delta \leqslant C_R e^{-\gamma t}\, \|\Psi\|'_{(R)}, \qquad t \geqslant 0. \tag{8.9}$$

Proof. We apply the standard Cook method: see, e.g., ref. 19, Thm. XI.4. Fix $\Psi \in \mathscr{H}'_{(R)}$ and define $W\Psi$, formally, as

$$W\Psi = \lim_{t \to \infty} U'_0(-t)\, U'(t)\, \Psi = U'_0(-T_1)\, U'(T_1)\, \Psi + \int_{T_1}^{\infty} \frac{d}{dt} U'_0(-t)\, U'(t)\, \Psi\, dt \tag{8.10}$$

with an appropriate $T_1 > 0$. We have to prove the convergence of the integral in the norm of the space $\mathscr{H}'_\delta$. First, observe that

$$\frac{d}{dt} U'_0(t)\,\Psi = \mathscr{A}'_0 U'_0(t)\,\Psi, \qquad \frac{d}{dt} U'(t)\,\Psi = \mathscr{A}' U'(t)\,\Psi,$$

where $\mathscr{A}'_0$ and $\mathscr{A}'$ are the generators to the groups $U'_0(t)$, $U'(t)$, respectively. Similarly to (5.4), we have

$$\mathscr{A}' = \begin{pmatrix} 0 & A \\ 1 & 0 \end{pmatrix}, \tag{8.11}$$

where $Au = \sum \partial_i (a_{ij}(x) \partial_j u) - a_0(x) u$. Therefore,

$$\frac{d}{dt} U_0'(-t) U'(t) \Psi = U_0'(-t)(\mathscr{A}' - \mathscr{A}_0') U'(t) \Psi. \tag{8.12}$$

(8.11) and (5.4) imply that

$$\mathscr{A}' - \mathscr{A}_0' = \begin{pmatrix} 0 & A - A_0 \\ 0 & 0 \end{pmatrix}. \tag{8.13}$$

Observe that $A - A_0 = \sum \partial_i b_{ij}(x) \partial_j - a_0(x)$, where $b_{ij}(x),\ a_0(x) \in C_0^\infty(B_{R_0})$ with some $R_0 > 0$, according to E1. Therefore, by (8.3), we have that

$$\begin{aligned} &|||U_0'(-t)(\mathscr{A}' - \mathscr{A}_0') U'(t) \Psi|||_\delta' \\ &\quad \leqslant C e^{\delta t} |||(\mathscr{A}' - \mathscr{A}_0') U'(t) \Psi|||_\delta' \\ &\quad \leqslant C e^{\delta t} \|((\mathscr{A}' - \mathscr{A}_0') U'(t) \Psi)^0\|_{H^{-1}(B_{R_0})} \\ &\quad \leqslant C e^{\delta t} \|(U'(t) \Psi)^1\|_{H^1(B_{R_0})}, \qquad t \geqslant 0. \end{aligned} \tag{8.14}$$

Then (8.12) and (8.7) imply, for $t \geqslant T = T(R, R_0)$, that

$$\left|\left|\left| \frac{d}{dt} U_0'(-t) U'(t) \Psi \right|\right|\right|_\delta' \leqslant C(R, R_0) e^{\delta t} e^{-\alpha t} \|\Psi\|_{(R)}' = C(R) e^{-\beta t} \|\Psi\|_{(R)}', \tag{8.15}$$

where $\beta = \alpha - \delta$. Choose $\delta > 0$ sufficiently small: $\delta < \alpha$. Then we have $\beta > 0$, and (8.15) implies

$$\int_T^{+\infty} \left|\left|\left| \frac{d}{dt} U_0'(-t) U'(t) \Psi \right|\right|\right|_\delta' dt \leqslant C_1(R) \|\Psi\|_{(R)}' < \infty.$$

Therefore, the existence of the limit in (8.10) follows if we choose $T_1 = T$. Furthermore, the operator $W: \mathscr{H}' \to \mathscr{H}_\delta'$ is continuous, and (8.10), (8.15) imply

$$|||(U_0'(-t) U'(t) - W) \Psi|||_\delta' \leqslant C_2(R) e^{-\beta t} \|\Psi\|_{(R)}', \qquad t \geqslant T. \tag{8.16}$$

Let us now choose $\delta < \alpha/2$. Then $\delta < \beta = \alpha - \delta$ and $\gamma = \beta - \delta = \alpha - 2\delta > 0$. Finally, set $r(t) \Psi := U'(t) \Psi - U_0'(t) W\Psi$, then (8.16) and (8.3) imply

$$\begin{aligned} |||r(t) \Psi|||_\delta' &= |||(U'(t) - U_0'(t) W) \Psi|||_\delta' = |||U_0'(t)(U_0'(-t) U'(t) - W) \Psi|||_\delta' \\ &\leqslant C_3(R) e^{\delta t} |||(U_0'(-t) U'(t) - W) \Psi|||_\delta' \\ &\leqslant C_4(R) e^{-\gamma t} \|\Psi\|_{(R)}', \qquad t \geqslant T. \quad \blacksquare \end{aligned}$$

Proof of Theorem 8.1. Equation (8.8) implies that for $Y_0 \in \mathscr{H}$ and $\Psi \in \mathscr{H}'_{(R)}$, for any $R > 0$,

$$\langle U(t)\, Y_0, \Psi \rangle = \langle U_0(t)\, Y_0, W\Psi \rangle + \langle Y_0, r(t)\, \Psi \rangle, \qquad t \geqslant 0 \tag{8.17}$$

By Proposition 8.8, operators $W_R \equiv W|_{\mathscr{H}'_{(R)}}$ and $r_R(t) \equiv r(t)|_{\mathscr{H}'_{(R)}}$ are continuous as maps $\mathscr{H}'_{(R)} \to \mathscr{H}'_\delta$. Therefore, the reflexivity of the Hilbert spaces implies the existence of the adjoint continuous operators $\Theta_R = W'_R$: $\mathscr{H}_\delta \to \mathscr{H}_{(R)}$ and $\rho_R(t) = r'_R(t)$: $\mathscr{H}_\delta \to \mathscr{H}_{(R)}$ for any $R > 0$. It remains to define $(\Theta Y_0)|_{B_R} = \Theta_R Y_0$ and $(\rho(t)\, Y_0)|_{B_R} = \rho_R(t)\, Y_0$ for any $R > 0$. ▮

9. CONVERGENCE TO EQUILIBRIUM FOR VARIABLE COEFFICIENTS

We deduce Theorem A from next two Propositions 9.1 and 9.2 (cf. Propositions 3.1 and 3.2).

Proposition 9.1. Family of measures $\{\mu_t, t \in \mathbb{R}\}$, is weakly compact in $\mathscr{H}^{-\varepsilon}$, $\forall \varepsilon > 0$.

Proposition 9.2. For every $\Psi \in \mathscr{D}$,

$$\hat{\mu}_t(\Psi) \equiv \int \exp(i\langle Y, \Psi \rangle)\, \mu_t(dY) \to \exp\{-\tfrac{1}{2}\, \mathscr{Q}_\infty(\Psi, \Psi)\}, \qquad t \to \infty. \tag{9.1}$$

Proof of Proposition 9.1. Similarly to Proposition 3.1, Proposition 9.1 follows from the bounds

$$\sup_{t \geqslant 0} E\, \|U(t)\, Y_0\|_R^2 < \infty, \qquad R > 0. \tag{9.2}$$

Theorem 8.1 implies that

$$\begin{aligned} E\, \|U(t)\, Y_0\|_R^2 &\leqslant 2E\, \|\Theta U_0(t)\, Y_0\|_R^2 + 2E\, \|r(t)\, Y_0\|_R^2 \\ &\leqslant C_1(R)\, E\, |||U_0(t)\, Y_0|||_\delta^2 + C_2(R)\, e^{-2\gamma\, |t|} E\, |||Y_0|||_\delta^2. \end{aligned}$$

Then (9.2) follows from (4.10) and (2.13). ▮

Proof of Proposition 9.2. Let $\Psi \in \mathscr{H}'_{(R)}$. Then Theorem 8.1 implies that

$$\begin{aligned} \hat{\mu}_t(\Psi) \equiv E \exp\{i\langle U(t)\, Y_0, \Psi \rangle\} &= E \exp\{i\langle \Theta U_0(t)\, Y_0 + r(t)\, Y_0, \Psi \rangle\} \\ &= E \exp\{i\langle \Theta U_0(t)\, Y_0, \Psi \rangle\} + v(t), \end{aligned} \tag{9.3}$$

where

$$v(t) = E[\exp\{i\langle \Theta U_0(t)\, Y_0, \Psi\rangle\}(\exp\{i\langle r(t)\, Y_0, \Psi\rangle\} - 1)].$$

Note that $v(t)$ vanishes as $t \to \infty$. In fact, Theorem 8.1 implies as above,

$$|v(t)| \leqslant E\,|\langle r(t)\, Y_0, \Psi\rangle| \leqslant C(R)\, \|\Psi\|'_{(R)}\, E\, \|r(t)\, Y_0\|_R \to 0, \qquad t \to \infty. \tag{9.4}$$

Finally, Proposition 3.2 and Corollary 4.3 imply that

$$E \exp\{i\langle \Theta U_0(t)\, Y_0, \Psi\rangle\} = E \exp\{i\langle U_0(t)\, Y_0, W\Psi\rangle\} \to \exp\{-\tfrac{1}{2}\, \mathscr{Q}_\infty(W\Psi, W\Psi)\}, \qquad t \to \infty, \tag{9.5}$$

and (9.3)–(9.5) imply (9.1). ▮

10. VAINBERG'S ESTIMATES

In this section we prove Lemma 8.7.

Proposition 10.1. Let E1–E3 hold, and $n \geqslant 3$ be odd. Then $\forall R$, $R_0 > 0$ there exist constants $\alpha, C(R, R_0) > 0$ and $T = T(R, R_0) > 0$ such that for $Y_0 \in \mathscr{H}$ with supp $Y_0 \subset B_{R_0}$,

$$\left\| \frac{\partial^k}{\partial t^k} U(t)\, Y_0 \right\|_R \leqslant C(R, R_0)\, e^{-\alpha t}\, \|Y_0\|_{R_0}, \qquad t \geqslant T, \quad k = 0, 1,... \tag{10.1}$$

Proof. Conditions E1–E3 allow us to use Theorem X.4 from ref. 12 which implies an asymptotic expansion for the solution $Y(\cdot, t) = U(t)\, Y_0$,

$$Y(x, t) = \sum_{j=1}^{N} \sum_{l=0}^{d_j} t^l e^{-i\omega_j t} Y_{jl}(x) + Z_N(x, t). \tag{10.2}$$

Here Im ω_j is a nonincreasing sequence, Im $\omega_j \to -\infty$, $d_j < \infty$, and the remainder $Z_N(x, t)$ satisfies the bounds (10.1) when Im $\omega_{N+1} < 0$. It remains to prove that all terms with Im $\omega_N \geqslant 0$ vanish. Assumptions E1 and E2 provide an a priori bound for finite energy solutions. Therefore, in accordance with Theorem 8 (or Lemma 10) of ref. 12, Chap. X, all increasing terms with Im $\omega_j > 0$ or with Im $\omega_j = 0$ and $l > 0$ vanish. By the same reason, each amplitude $Y_{jl}(x)$ with Im $\omega_j = 0$ and $l = 0$ has a finite global energy and hence is an eigenfunction of the operator L (see (1.1)) with the eigenvalue $-\omega_j^2$. Therefore, an extension of Kato's Theorem[23] Thm. 2.1 implies the absence of a discrete spectrum inside the continuous spectrum, i.e., $Y_{jl}(x)=0$ if $\omega_j\neq 0$. Amplitude with $\omega_j=0$ and $l=0$ vanishes since the operator L is strictly negative. ▮

Therefore, we get by duality the following bounds for $\Psi\in\mathscr{H}'_{(R)}$:

$$\left\|\frac{\partial^k}{\partial t^k}U'(t)\,\Psi\right\|_{H^{-1}(B_{R_0})\oplus L^2(B_{R_0})}\leqslant C(R,R_0)\,e^{-\alpha t}\,\|\Psi\|'_{(R)},\qquad t\geqslant T,\quad k=0,1,... \tag{10.3}$$

Recall that (5.2) implies the representation $U'(t)\,\Psi=(\dot{\psi}(\cdot,t),\psi(\cdot,t))$ where $\psi(x,t)$ is a solution to $\ddot{\psi}=A\psi$. Then (10.3) with $k=0$ implies

$$\|\psi(\cdot,t)\|_{L^2(B_{R_0})}\leqslant C(R,R_0)\,e^{-\alpha t}\,\|\Psi\|'_{(R)},\qquad t\geqslant T. \tag{10.4}$$

Similarly, (10.3) with $k=1$ implies that

$$\|\dot{\psi}(\cdot,t)\|_{L^2(B_{R_0})}\leqslant C(R,R_0)\,e^{-\alpha t}\,\|\Psi\|'_{(R)},\qquad t\geqslant T. \tag{10.5}$$

Also, by virtue of $\ddot{\psi}=A\psi$, we get

$$\|A\psi(\cdot,t)\|_{H^{-1}(B_{R_0})}\leqslant C(R,R_0)\,e^{-\alpha t}\,\|\Psi\|'_{(R)},\qquad t\geqslant T. \tag{10.6}$$

Note that (10.5) is a part of the bound (8.7). It remains to obtain the bound for $\|\psi(\cdot,t)\|_{H^1(B_{R_0})}$. We deduce this bound from the interior Schauder estimates[12] Thm. VI.5 for an elliptic operator A:

$$\|\psi(\cdot,t)\|_{H^1(B_{R_0})}\leqslant C(\|A\psi(\cdot,t)\|_{H^{-1}(B_{R_0+1})}+\|\psi(\cdot,t)\|_{L^2(B_{R_0+1})}). \tag{10.7}$$

We use (10.4) and (10.6) with R_0+1 instead of R_0 in the right hand side of (10.7) and obtain

$$\|\psi(\cdot,t)\|_{H^1(B_{R_0})}\leqslant C_1(R,R_0)\,e^{-\alpha t}\,\|\Psi\|'_{(R)},\qquad t\geqslant T(R,R_0+1). \tag{10.8}$$

Therefore, (10.5), (10.8) imply (8.7). ▮

APPENDIX A. FOURIER TRANSFORM CALCULATIONS

We consider dynamics and CFs of the solutions to the system (3.2). Let $F\colon w\mapsto\hat{w}$ denote the FT of a tempered distribution $w\in S'(\mathbb{R}^n)$ (see, e.g., ref. 18). We also use this notation for vector- and matrix-valued functions.

A.1. Dynamics in the Fourier Space

In the Fourier representation, the system (3.2) becomes $\dot{\hat{Y}}(k,t) = \hat{\mathscr{A}}_0(k)\,\hat{Y}(k,t)$, hence

$$\hat{Y}(k,t) = \hat{\mathscr{G}}_t(k)\,\hat{Y}_0(k), \qquad \hat{\mathscr{G}}_t(k) = \exp(\hat{\mathscr{A}}_0(k)\,t). \tag{A.1}$$

Here we denote

$$\hat{\mathscr{A}}_0(k) = \begin{pmatrix} 0 & 1 \\ -|k|^2 & 0 \end{pmatrix}, \qquad \hat{\mathscr{G}}_t(k) = \begin{pmatrix} \cos|k|\,t & \dfrac{\sin|k|\,t}{|k|} \\ -|k|\sin|k|\,t & \cos|k|\,t \end{pmatrix}. \tag{A.2}$$

A.2. Cvariance Functions in Fourier Space

Translation invariance (1.4) implies that in the sense of distributions

$$E(\hat{Y}_0(k) \otimes_C \hat{Y}_0(k')) = F_{x\to k}F_{y\to k'}q_0(x-y) = (2\pi)^n\,\delta(k+k')\,\hat{q}_0(k), \tag{A.3}$$

where $\otimes_C$ stands for tensor product of complex vectors. Now (A.1) and (A.2) give in the matrix notation,

$$E(\hat{Y}(k,t) \otimes_C \hat{Y}(k',t)) = (2\pi)^n\,\delta(k+k')\,\hat{\mathscr{G}}_t(k)\,\hat{q}_0(k)\,\hat{\mathscr{G}}'_t(k). \tag{A.4}$$

Therefore,

$$q_t(x-y) := E(Y(x,t) \otimes_C Y(y,t)) = F^{-1}\hat{\mathscr{G}}_t(k)\,\hat{q}_0(k)\,\hat{\mathscr{G}}'_t(k). \tag{A.5}$$

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